Equivalent Environments and Covering Spaces for Robots

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Abstract

This paper formally defines a robot system, including its sensing and actuation components and its surrounding environment, as a general, topological dynamical system. The focus is on determining general conditions under which various environments in which the robot can be placed are indistinguishable, from the viewpoint or experience of the robot. An envronment is defined as a tuple that includes a Polish state space with which 'interaction' occurs through a sensorimotor structure that is defined in terms of a Borel sensor mapping and a Polish space of control signals that acts on the state space. A key result is that, under very general conditions, covering maps witness such indistinguishability. This formalizes the intuition behind the well studied loop closure problem in robotics. An important special case is where the sensor mapping reports an invariant of the local topological (metric) structure of an environment because such structure is preserved by (metric) covering maps. Whereas coverings provide a sufficient condition for the equivalence of environments, we also give a necessary condition using bisimulation. The overall framework is applied to unify previously identified phenomena in robotics and related fields, in which moving agents with sensors must make inferences about their environments based on limited data. Many open problems are identified.

Contents

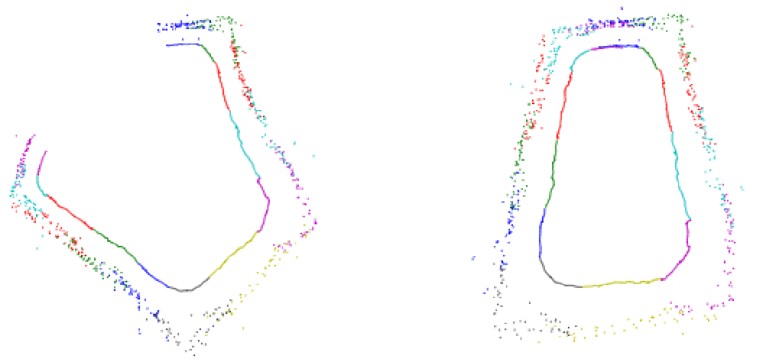

Figure 1: Map reconstruction before loop closure (left) and after loop closure (right). Note how a new metric has been defined on the space to support the new information that the distance of two previously distinct points equals now to zero. Figure reproduced from [35].

1 Introduction

When a mobile robot explores a multiply connected environment using sensors, it frequently encounters the well-known problem of *loop closure*, in which it must detect that it has returned to a previously visited location. Robots combine information from sensor observations at multiple times, leading to the *filtering* problem (also known as *sensor fusion*) of appropriately aggregating or summarizing its data. Shown in Figure 1, the robot becomes mistaken (from our perspective) about its position over time as it tries to build a geometric representation of its environment. The problem is part of *SLAM (simultaneous localization and mapping)* [6, 15], a critical operation in many robots and autonomous systems. Similarly, a person donning a VR headset can be tricked into walking in circles when they believe they are heading straight, using a method known as *redirected walking* [25]. Likewise, according to a popular theory [24], a moth inadvertently travels in circles around a fire because it cannot distinguish the fire and a celestial light source[1]. This phenomenon occurs, and can be studied, in the framework of minimalist robotics such as Braitenberg robots [1], wall-following robots [18, 26], and robots with topological sensing and filtering, such as the gap-navigating robots of [30]. It is also closely related to some version of the graph exploration problem [20, 27].

All of these examples (and many more) share an underlying principle: The covering map $f\colon \mathbb{R} \to S^1$, $t \mapsto e^{it}$ commutes with the agent's sensorimotor behavior, whether the agent be a robot, human, or insect. More generally, topological ambiguities arise from limited sensing and actuation capabilities, and we show in this paper that they are naturally understood in terms of covering spaces. Figure 2 shows examples of spaces and their coverings which, considered as robot's environments, will be shown to be strongly indistinguishable (Section 6.3). As a converse we obtain an invariant of the

[1]This theory was recently disputed in [9].

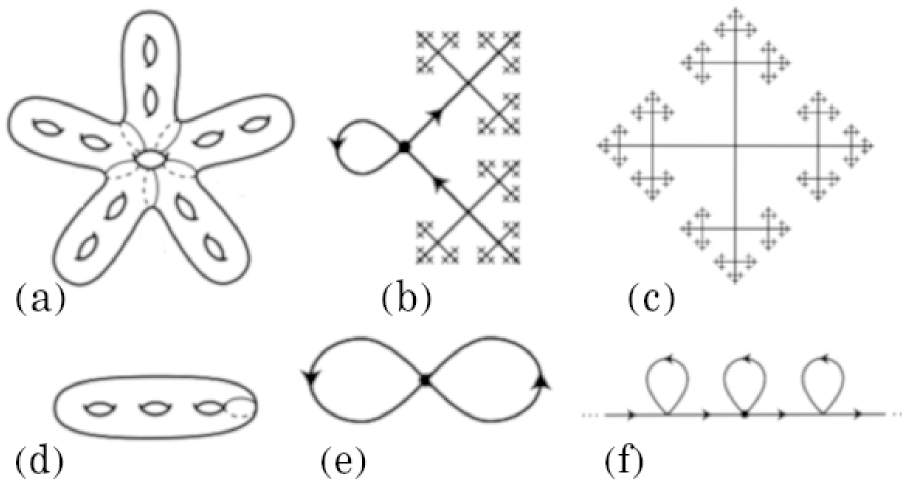

Figure 2: The spaces on the bottom are strongly indistinguishable from the ones above them. The 2-manifold (a) is a covering space of (d), the 1-complex (b) is a covering space of (e) and (f), whereas (c) is the universal covering space of (b), (e), and (f). For a robot, which senses the local homeomorphism type, (b), (c), (e), and (f) are mutually (not strongly) indistinguishable. Figures are reproduced from [16].

Figure 3: A robot, which senses the local homeomorphism type, can distinguish between these spaces, which implies that they cannot have a common covering space (see Example 6.10). This constitutes an application of the present theory of mathematical robotics to topology.

purely topological equivalence relation between spaces which holds if and only if they have a common covering space, see Figure 3.

We approach this through the abstract theory of information spaces which was originally proposed and developed in [21, 31] and extended recently in [23, 34]. In it, the robot (or more broadly, an agent), has access exclusively to the sequence of sensorimotor interactions with the environment given by a sequence $\eta = (u_0, y_0, u_1, y_1, \ldots, u_n, y_{n-1})$ in which u_k is the motor input at stage k and y_k is the sensory data at stage k. These sequences are called *history information states*. The robot may be thought of exploring a tree of all possible history information states ordered by end-extension. This tree is called the *history information space*.

We begin by reviewing prior work on loop closure, SLAM, graph exploration, and minimal filtering in Section 2. In Section 3 we develop the general theory of the spaces of control signals, trajectories, and what we call *path actions*. We review some prior work where such spaces have been defined (Sections 3.1-3.2) and strengthen some of the results of [36] in order to motivate the more general topological definitions in Section 3.3. Our main result in this section is Corollary 3.15 which says that the map assigning robot's trajectories to control signals is continuous under the assumption that the mere position of the robot is a continuous function of the control signals. We prove this under very general topological assumptions on the set of control signals (Definition 3.10).

Then, in Section 4 we apply the framework developed in 3-3.3 to define the notion of *sensorimotor structure* as well as the general notion of *environment*. The sensorimotor structure consists of a sensor mapping h and a path action p which define the coupling between the ambient space and the robot's sensors and actuators (Section 4.1). Then, we define the continuous version of history information state space -4.2) generalizing the discrete time version of [21].

We then use these tools to define a class of indistinguishability equivalence relations on environments in Section 5 and prove a number of basic results about them. These are interesting from the perspective of both robotics and topology. The most basic one, which we also analyze the most, says that two environments are indistinguishable, if any control signal (no matter how long) will yield identical sensory readings in both of them (Definitions 5.1 and 5.7).

In Section 6 we introduce the idea of covering spaces in this context and prove our next main results (Theorem 6.5, Corollary 6.7, and Theorem 6.9) which formalize the idea that a covering map enables lifting the sensorimotor structure to make the covered and covering spaces indistinguishable, or that environments which have a common covering (or are both coverings of an) environment, then they are also indistinguishable.

In the end of the paper we prove our final main result, Theorem 7.4, giving a complete characterization of the $\equiv$-equivalence in terms of bisimulations. Section 8 then revisits the examples of Section 2. Finally we prove a general result about robots whose sensors report an isometry-invariant of their local neighbourhood, Theorem 8.7.

2 Closing the Loop via Sensing and Filtering

This section reviews the well-known loop closure problem in robotics, both in the context of engineering (Section 2.1), and as analyzed in the theoretical minimalist setup (Section 2.2). Finally, we touch upon the related topic of graph exploration in Section 2.3.

2.1 Loop closure in SLAM

In simultaneous localization and mapping (SLAM) problems, robots are moving in their environments while also scanning them. They may also keep track of various non-visual parameters such as data from their accelerometer, GPS-coordinates, wheel encoders, sonars, lasers and so on [32]. Once the robot circles back to an area where it has been before, it is incumbent on the robot to be capable of "meeting the ends", that is, identifying the place as one which has been already visited and applying this knowledge to the map reconstruction problem. The *visual loop closure* is the subcategory of loop closure problems in which this recognition has to be made based on visual sensors alone (possibly including 3D-sensors such as lasers). Even when motion detection data is available, accumulated errors may make loop closure detection difficult based on path integration methods. This error makes the loop closure problem in essence topological, as opposed to metric. In fact, once two places in the internally reconstructed *metric* maps have been identified as the same, usually a new metric has to be defined on the map to make it correct, see Figure 1

In many applications the visual sensors are powerful monocular or binocular multi-megapixel cameras and laser measurements [2, 13]. Many computational methods have been developed to perform this highly complex task. Methods which work directly with data include statistical such as Bayesian inference [2], hybrid statistical-geometric such as direct sparse odometry [8, 12], graph based methods such as local registration and global correlation [14] and other sophisticated methods such as ORB-SLAM [22] and graph-matching with deep learning [7]. These include scan matching, consistent pose estimation, map correlation, clustering, and dimension analysis. There is also a class of methods which are based on image recognition where the visual images are labeled using neural networks and then the algorithms operate with those labels.

2.2 Minimal filtering and minimal SLAM

At the other extreme, minimalist models which analyze the logical and geometric underpinnings of SLAM have been proposed [18, 26, 29, 30]. In this line of work the analysis is performed in mathematically well-defined environments and the models assume only simple feature detectors. The robots are often capable of building very minimalist, topological representations of the environment so that the required tasks are accomplished reliably and provably. Loop closure features in this theory too. It can be achieved by the robot dropping recognizable pebbles in the environment at selected locations [29, 30]. The framework of minimalist robotics has the theoretical advantage that it enables one to reason about the limiting cases pertaining in particular to whether loop closure is theoretically possible or not.

Consider gap-navigation trees introduced in [30]. Here, the robot is moving in a planar environment $O \subset \mathbb{R}^2$ and is equipped with a sensor which only reports the directions in which the distance-to-the-boundary function has discontinuities. These correspond to corners and turns in the boundary of the environment. This sensor can be thought of as detecting a topological invariant of the star-convex environment of the robot's current location (Figure 4). We will make this statement precise in Example 8.1. Using this data the robot is able to internally build a tree-like model which encodes the convex region structure sufficiently well for the robot to be able to optimally solve navigation tasks in a simply connected environment. This setup is subject to the loop-closure problem. If the environment is not simply connected, the robot may start going around a circular obstacle, but updating the model as if it is seeing new regions ad infinitum. It was shown in [30] that their model will fail in non-simply connected environments. We show in Example 8.1 the same thing within our new framework. By equipping the robot with loop closure detection (using pebbles), the authors of [30] show that then the navigation problem can be solved in non-simply

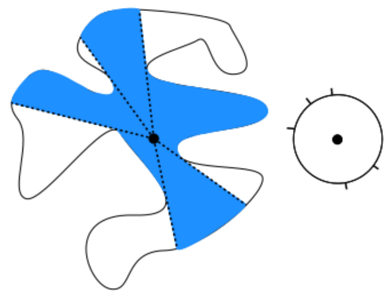

Figure 4: The visible star-convex region (left) and the outcome of sensor filtering (right). The filter is a topological invariant of the star-region as we will see in Section 8. Image reproduced from [30]

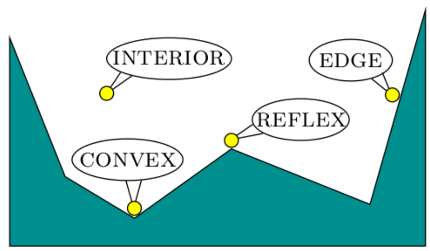

Figure 5: An example of a sensor mapping which is a metric invariant of the local neighbourhood of the robot. See Section 8 for more details. Image reproduced from [18]

connected environments too.

In [18] the authors analyze a simple robot with local sensors that moves in an unknown polygonal environment. The robot is capable of sensing local geometric structure of the environment: it can detect whether or not it is on the boundary and if it is, whether or not it is on a vertex and if it is, whether it is a convex or a reflex (concave) vertex (Figure 5). The robot can also leave pebbles in the environment which it can later detect. Using this machinery the robot is shown to be capable of various tasks. Our interest in this example is that here also, the sensor is a geometric (in this case metric, not topological) invariant of the local neighbourhood of the robot, see Section 8.

In [28] the authors "analyze a problem in which an unpredictable moving body travels among obstacles and binary detection beams. The task is to determine the possible body path based only on the binary sensor data. This is a basic filtering problem encountered in many settings, which may arise from physical sensor beams or virtual beams that are derived from other sensing modalities." (quote from [28], see also Figure 6) The authors show among other things that if the region is partitioned by the beams into simply connected regions, then the body can know the homotopy type of the traversed trajectory.

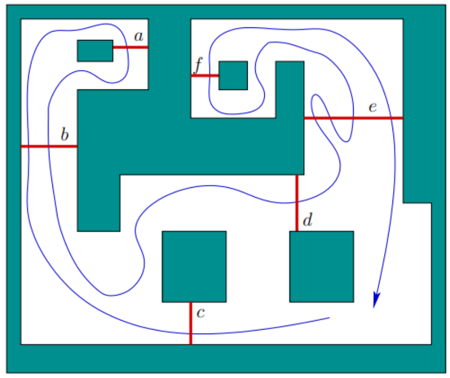

Figure 6: Navigation using only the data from the beams. Image reproduced from [28]

2.3 Graph exploration

Graph exploration is a family of problems in the intersection of graph theory and theoretical robotics. A subcategory of them ask the robot to generate a map of the environment, either a complete one or a sufficient one for its tasks [20, 27].

This set of problems must include some form of loop closure, for otherwise a robot cannot distinguish between the graphs (b), (c), (e), and (f) shown on Figure 2. The theory outlined in this paper applies to this context in the limited special case when the robot is not allowed to make marks in the environment.

3 Basic Models and Theory Development

In this section we introduce the basic setup. In Section 3.1 we introduce the space $\mathcal{U}_M$ of measurable control signals, and in Section 3.2 the space $\mathcal{X}$ of continuous trajectories that robot traverses in X. We analyze, building on [36] the case when X is a differential manifold and define the functions r and $\bar{r}$ which connect the control signal to the resulting configuration and the trajectory of the robot respectively. Using the results of 3.1 and 3.2, we then formulate appropriate generalizations to a topological framework in Section 3.3. There, we define $\mathcal{U}$ which is a generalization of $\mathcal{U}_M$, and the path action p (and the induced $\bar{p}$) which is a generalization of r (respectively of $\bar{r}$). The space X is merely generalized from being a differential manifold to being a Polish space without any changes in the definition of $\mathcal{X}$.

3.1 Space of control signals

This section will follow [36] in defining the space $\mathcal{U}$ of control signals. Later, in Section 3.3, we will have a bit more general definition. The robot has an arbitrary, nonempty set of inputs U which in this section is a topological space. A *control signal* is a measurable function $\bar{u}\colon [0,T) \to U$. We use half-open intervals as the domain (deviating from [36]) so that we can naturally concatenate them. Over its lifetime, the robot computer or controller generates a signal $\bar{x}\colon \mathbb{R}_{\geqslant 0} \to U$, which can be obtained as a concatenation of bounded-time signals. This will be made formal.

Definition 3.1. Let $\mathcal{U}_M$ be the set of all measurable functions $\bar{u}\colon [0,T) \to U$ for $T \in \mathbb{R}_{\geqslant 0}$. For $T = 0$ this is the empty function $\bar{u} = \varnothing$. Denote by $T = |\bar{u}|$ the supremum of the domain of $\bar{u}$. For any two elements $\bar{u}_1, \bar{u}_2 \in \mathcal{U}_M$, let $\bar{u}_1 \oplus \bar{u}_2$ be the control signal $\bar{u}$ with $|\bar{u}| = |\bar{u}_1| + |\bar{u}_2|$ defined by

$$\bar{u}(t) = \begin{cases} \bar{u}_1(t) & \text{if } t < |\bar{u}_1| \\ \bar{u}_2(t - |\bar{u}_1|) & \text{otherwise.} \end{cases}$$

Given $\bar{u} \in \mathcal{U}_M$, denote by $\bar{u} \restriction [0,t)$ the restriction of $\bar{u}$ to $[0,t)$. This can be shortened to $\bar{u} \restriction t$ or to $\bar{u}_{<t}$. If $t \geqslant |\bar{u}|$, then $\bar{u} \restriction t = \bar{u}$. Let $\bar{u}_1 \lhd \bar{u}_2$ denote that $|\bar{u}_1| < |\bar{u}_2|$ and $\bar{u}_2 \restriction |\bar{u}_1| = \bar{u}_1$. Given $\bar{u} \in \mathcal{U}_M$ and $t \in \mathbb{R}_{\geqslant 0}$ we define $\bar{u}_{<t}$ and $\bar{u}_{\geqslant t}$ to be the unique two elements of $\mathcal{U}_M$ with the property that $|\bar{u}_{<t}| = t$ and $\bar{u} = \bar{u}_{<t} \oplus \bar{u}_{\geqslant t}$. ⊣

Proposition 3.2. *$\mathcal{U}_M$ is*

(1) *closed under $\oplus$, meaning that for all $\bar{u}_0, \bar{u}_1 \in \mathcal{U}_M$ we have $\bar{u}_0 \oplus \bar{u}_1 \in \mathcal{U}_M$,*

(2) closed under segmentation, *meaning that for all $\bar{u}$ and all t, also $\bar{u}_{<t}$ and $\bar{u}_{\geqslant t}$ are in $\mathcal{U}_M$, and*

(3) extensive, *meaning that for all $\bar{u} \in \mathcal{U}_M$, there is $\bar{u}_1 \in \mathcal{U}_M$ such that $\bar{u} \lhd \bar{u}_1$ and $|\bar{u}_1| \geqslant |\bar{u}| + 1$.*

Proof. These all follow easily from the fact that $\mathcal{U}_M$ is the collection of all measurable functions □

Remark 3.3.

1. As is standard in functional analysis, we actually consider equivalence classes of measurable functions that differ on a set of measure zero, without altering notation.

2. The ordering $\lhd$ is a tree order on $\mathcal{U}_M$, meaning $\lhd$-incompatible control signals have no common $\lhd$-extensions.

3. We always have $\bar{u}_1 \lhd \bar{u}_1 \oplus \bar{u}_2$.

4. Also, $(\bar{u}_0 \oplus \bar{u}_1)_{<|u_0|} = u_0$ and $(\bar{u}_0 \oplus \bar{u}_1)_{\geqslant |u_0|} = u_1$.

5. $\bar{u}_{<t}$ is the same as the restriction $\bar{u} \restriction [0,t)$.

6. $\bar{u}_{\geqslant t}$ is the final segment of $\bar{u}$ which can be explicitly defined as a function with domain $[0, t_0)$ where $t_0 = |\bar{u}| - t$ and for all $0 \leqslant t_1 < t_0$ we have $\bar{u}_{\geqslant t}(t_1) = \bar{u}(t + t_1)$. ⊣

Following [36] we define the L_1-type metric on $\mathcal{U}_M$.

Definition 3.4 (Metric on $\mathcal{U}_M$)**.** Assume that d_U is a metric on U. Then, given measurable $\bar{u}_1\colon [0,T_1)\to U$ and $\bar{u}_2\colon [0,T_2)\to U$, define

$$\varrho_{\mathcal{U}_M}(\bar{u}_1,\bar{u}_2)=\int_0^T d_U(\bar{u}_1(t),\bar{u}_2(t))dt+|T_1-T_2|,$$

where $T=\min\{T_1,T_2\}$. $\dashv$

A metric is called *Polish* if it makes the space complete and separable.

Proposition 3.5. *If d_U is bounded, then $\varrho_{\mathcal{U}_M}$ is a Polish metric on $\mathcal{U}_M$.*

Proof. That it is a metric was already observed in [36]. A dense countable set was also constructed in [36]. Suppose $(\bar{u}_n)$ is a Cauchy sequence. Because of the second term in the definition of the metric, $|\bar{u}_n|$ must converge to some T. Let (T_n) be an increasing sequence converging to T with the property that for all $m\geqslant n$ we have $|\bar{u}_m|>T_n$. Then for each n, the sequence $(\bar{u}_m\restriction_{<T_n})_{m\geqslant n}$ is a Cauchy sequence in the classical L_1-metric on the space of all continuous functions from $[0,T_n)$ into U which is known to be complete. Thus, let $\bar{u}_n^*$ be the limit of that sequence. Then, observe that $\bar{u}_n^*\lhd\bar{u}_{n+1}^*$ for all n, and $\bigcup_n\bar{u}_n^*$ is the limit of $(\bar{u}_n)$ in $\mathcal{U}_M$. $\square$

The following acts as a motivation for the generalization in Definition 3.10.

Proposition 3.6. *For $\bar{u}\in\mathcal{U}_M$ and $t\in\mathbb{R}_{\geqslant 0}$, the following functions are continuous:*

(1) $\tau_1\colon\bar{u}\mapsto|\bar{u}|$*,*

(2) $\tau_2\colon(\bar{u},t)\mapsto\bar{u}_{<t}$*.*

Proof. (1) Suppose $s<|\bar{u}|<t$ and let $\varepsilon=\min\{|\bar{u}|-s,t-|\bar{u}|\}$. Since $|\bar{u}-\bar{u}_1|$ is bounded by $\varrho_{\mathcal{U}_M}(\bar{u}_1,\bar{u})$ for all $\bar{u}_1$, if $\varrho_{\mathcal{U}_M}(\bar{u}_1,\bar{u})<\varepsilon$, then also $s<|\bar{u}_1|<t$. This shows that the inverse image of the open interval (s,t) under τ_1 is open.

(2) Let $\mathcal{U}_M\times\mathbb{R}_{\geqslant 0}$ be equipped with the metric $\delta((\bar{u},t),(\bar{v},s))=\varrho_{\mathcal{U}_M}(\bar{u},\bar{v})+|t-s|$. This metric is compatible with the product topology; thus, it suffices to show continuity of τ_2 w.r.t. δ. Suppose $(\bar{u},t),(\bar{v},s)\in\mathcal{U}_M\times\mathbb{R}_{\geqslant 0}$. Let $T_1=\min\{|\bar{u}|,|\bar{v}|,t,s\}$, $T_2=\min\{|\bar{u}|,|\bar{v}|\}$, $T_3=\min\{|\bar{u}|,t\}$, and $T_4=\min\{|\bar{v}|,s\}$. Now $T_1\leqslant T_2$, $T_3=|\bar{u}_{<t}|\leqslant|\bar{u}|+t$, $T_4=|\bar{v}_{<s}|\leqslant|\bar{v}|+s$, and so we get:

$$\begin{aligned}
\varrho_{\mathcal{U}_M}(\tau_2(\bar{u},t),\tau_2(\bar{v},s)) &= \varrho_{\mathcal{U}_M}(\bar{u}_{<t},\bar{v}_{<s})\\
&= \int_0^{T_1} d_U(\bar{u}_{<t}(z),\bar{v}_{<s}(z))dz+||\bar{u}_{<t}|-|\bar{v}_{<s}||\\
&= \int_0^{T_1} d_U(\bar{u}(z),\bar{v}(z))dz+|T_3-T_4|\\
&\leqslant \int_0^{T_2} d_U(\bar{u}(z),\bar{v}(z))dz+||\bar{u}|-|\bar{v}|+t-s|\\
&\leqslant \int_0^{T_2} d_U(\bar{u}(z),\bar{v}(z))dz+||\bar{u}|-|\bar{v}||+|t-s|\\
&= \varrho_{\mathcal{U}_M}(\bar{u},\bar{v})+|t-s|\\
&= \delta((\bar{u},t),(\bar{v},s)),
\end{aligned}$$

which shows that τ_2 is in fact 1-Lipschitz and therefore continuous. $\square$

3.2 Space of trajectories and path actions given by differential structures

We first clarify our notion of the state space and what is its relationship to the ambient space in which the robot is, and to the notion of an environment defined in Section 4.1. A robot, as a body occupying physical space, can be in various configurations. The robot is then embedded in some ambient space which both restricts and extends this set. On the one hand, not only can the body be in some configuration, but it can also be in different locations and have different orientations in the ambient space. On the other hand, the ambient space may restrict the range of possible configurations that the robot's body can achieve. This gives rise to the state space X. For example, consider a car-like robot with four wheels and front steering. Its body's configuration space is a subset of $(S^1)^4 \times \mathbb{R}$. Once embedded in an ambient 3D world $\mathbb{R}^3$ and restricting the car to contact a planar surface, the space of robot's possible states X becomes a subset of $(S^1)^4 \times \mathbb{R} \times \mathbb{R}^2 \times S^1$. The third and fourth factors account for the car's position and orientation, respective, in the plane. Thus, the *ambient space* is the theoretical space in which the robot is and we do not refer to it in our theory. Often, $x \in X$ may also encode configuration velocities and other environmental particulars. We assume in this paper that X is a metric space. We obtain the notion of an *environment* in the next section by equipping X with a sensor mapping and a path action.

A control signal $\bar{u}\colon [0,T) \to U$ influences the robot's *state* in the state space X. If X is a smooth manifold, as in many applications, we can consider a parameterized vector field $f\colon X \times U \to TX$, where TX is the tangent bundle such that $f(x,u) \in T_xX$. Then, each $\bar{u}$, given an initial point $x_0 \in X$, yields a trajectory $\bar{x}\colon [0,T] \to X$ is the integral curve satisfying the following:

$$\bar{x}(0) = x_0 \quad \bar{x}'(t) = f(\bar{u}, x(t)) \qquad \text{for all } t \in [0,T]$$

If f is continuous, then $\bar{x}$ will also be continuous. The existence and uniqueness of $\bar{x}$ for locally Lipschitz f follows from the Picard-Lindelöf theorem. Thus, the range of this map $\bar{u} \mapsto \bar{x}$ is included in the space of all continuous paths $\bar{x}\colon [0,T] \to X$. Motivated by this, we define:

$$\mathcal{X} = \bigcup_{T \in \mathbb{R}_{\geqslant 0}} C([0,T], X). \tag{1}$$

Again following [36], and assuming that X is equipped with the metric d_X, we can equip $\mathcal{X}$ with the metric

$$\varrho_{\mathcal{X}}(\bar{x}_1, \bar{x}_2) = \sup\{d_X(\bar{x}_1(t), \bar{x}_2(t)) \mid t \in [0,T]\} + |T_2 - T_1|, \tag{2}$$

where $T = \min\{T_1, T_2\}$. As with $\mathcal{U}_M$, we show that $\mathcal{X}$ is a Polish space.

Proposition 3.7. *If d_X is a Polish metric on X, then $\varrho_{\mathcal{X}}$ is a Polish metric on $\mathcal{X}$.*

Proof. The argument for completeness is similar to that of the proof of Proposition 3.5. The only difference is that instead of the L_1-metric we use the sup-metric which is also known to be complete (in fact Polish) in this case [19, (4.19)]. To find a countable dense set, again use the fact that $\mathcal{X}_T = \{\bar{x} \in \mathcal{X} : |\bar{x}| = T\}$ is Polish for all T and let

$$D = \bigcup_{T \in \mathbb{Q}_+} D_T,$$

where D_T is a dense countable set of $\mathcal{X}_T$ and T ranges over positive rationals. As a countable union of countable sets D is countable and is easily seen to be dense in $\mathcal{X}$. □

Denote by $\bar{r} \colon \mathcal{U}_M \times X \to \mathcal{X}$ the map that takes $(\bar{u}, x)$ to the corresponding trajectory $\bar{x}$,

$$\bar{r} \colon (\bar{u}, x) \mapsto \bar{x}. \tag{3}$$

Let $r \colon \mathcal{U}_M \times X \to X$ be the map $r(\bar{u}, x) = \bar{r}(\bar{u}, x)(|\bar{r}(\bar{u}, x)|)$. It was shown in [36] that if f is Lipschitz and X is a subspace of $\mathbb{R}^n$, then $\bar{r}$ is continuous. We prove a slight strengthening of that:

Proposition 3.8. *If f is uniformly continuous and $X \subset \mathbb{R}^n$, then both $\bar{r}$ and r are continuous.*

Remark 3.9. We will see in Corollary 3.15 that under very general conditions the continuity of r implies the continuity of $\bar{r}$. ⊣

Proof. We prove the continuity of $\bar{r}$. The continuity of r will then follow from the continuity of the projection map $\bar{x} \mapsto \bar{x}(|\bar{x}|)$. Fix $(\bar{u}_1, x_1)$ and $(\bar{u}_2, x_2)$ arbitrarily. Let $T_1 = |\bar{u}_1|$, $T_2 = |\bar{u}_2|$ and $T = \min\{T_1, T_2\}$. Let $\varepsilon > 0$ and δ_1 be chosen such that $d(f(x_2, u_2), f(x_1, u_1)) < \varepsilon/(3T)$ whenever $d_X(x_1, x_2) + d_U(u_1, u_2) < \delta_1$, which exists by the uniform continuity of f. Let $\delta = \min\{\varepsilon, \delta_1\}/3$. Suppose now $\varrho_{\mathcal{U}_M}(\bar{u}_1, \bar{u}_2) + |x_2 - x_1| < \delta$ and let $\bar{x}_1 = r(\bar{u}_1, x_1)$, $\bar{x}_2 = r(\bar{u}_2, x_2)$. Now,

$$\begin{aligned}
&\varrho_{\mathcal{X}}(\bar{x}_1, \bar{x}_2) \\
&\leqslant \sup\left\{|x_1 - x_0| + \int_0^t \big|f(\bar{x}_1(t), \bar{u}_1(t)) - f(\bar{x}_2(t), \bar{u}_2(t))\big| dt \mid t \in [0, T]\right\} + |T_2 - T_1| \\
&\leqslant \sup\left\{\int_0^t \varepsilon/(3T) dt \mid t \in [0, T]\right\} + |x_2 - x_1| + |T_2 - T_2| \\
&= \sup\left\{t\varepsilon/(3T) \mid t \in [0, T]\right\} + |x_2 - x_1| + |T_2 - T_2| \\
&= \varepsilon/3 + \delta + \delta \\
&\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon,
\end{aligned}$$

which proves that r is continuous at the arbitrary point $(\bar{u}_1, x_1)$. □

3.3 Topological versions

The purpose of this section is to free ourselves from differential calculus and enable more flexible usage of topological machinery. In Definition 3.1 we required that U is a topological space, because we had to talk about measurable functions with range U. In the below definition we do not need to equip U with a topology, although usually in most applications it has a natural topology that comes with it.

Definition 3.10. Suppose U is any set and let U^* be the set of *all* functions $\bar{u}\colon [0,T) \to U$. Define $|\bar{u}|$, $\lhd$ and $\oplus$ the same way as in Definition 3.1. We define $\mathcal{U}$ to be any subset of U^* which satisfies (2) and (3) of Proposition 3.2, that is, *closed under segmentation* and *extensive*, and that it is equipped with a topology satisfying Lemma 3.6, that is, the projection maps $\tau_1\colon \bar{u} \mapsto |\bar{u}|$ and $\tau_2\colon (\bar{u},t) \mapsto \bar{u}_{<t}$ are continuous. $\dashv$

We use Propositions 3.2 and 3.6 to justify Definition 3.10 as a proper generalization of $\mathcal{U}_M$ of the previous section. We will use the space $\mathcal{X}$ the way we already defined it, see (1) and (2), assuming X is a Polish metric space (we no longer assume that it is a manifold).

Finally, here is the definition of path action which is independent of a differential, or other non-topological, structure on X or U:

Definition 3.11. A *path action* of $\mathcal{U}$ on X is a continuous function $p\colon \mathcal{U} \times X \to X$ such that

(PA1) $p(\varnothing, x) = x$ for all $x \in X$, and

(PA2) $p(\bar{u}, x) = p(\bar{u}_{\geqslant t}, p(\bar{u}_{<t}, x))$ for all $x \in X$, all $\bar{u} \in \mathcal{U}$ and all $t \in \mathbb{R}_{\geqslant 0}$.

$\dashv$

Remark 3.12. Perhaps a more natural definition of an action would to have the following clause instead of $(PA2)$:

(PA2′) $p(\bar{u}_0 \oplus \bar{u}_1, x) = p(\bar{u}_1, p(\bar{u}_0, x))$ for all $x \in X$, all $\bar{u}_0, \bar{u}_1 \in \mathcal{U}$.

This is equivalent to (PA2) when $\mathcal{U}$ is closed under $\oplus$. The space $\mathcal{U}_M$ is such (Proposition 3.2). Suppose, however, one wanted to consider $\mathcal{U}_C \subset \mathcal{U}_M$ which consists only of continuous paths. Then, $\mathcal{U}_C$ is not closed under $\oplus$: if $\lim_{t\to|\bar{u}_0|} \bar{u}_0(t) \neq \bar{u}_1(0)$, then $\bar{u}_0 \oplus \bar{u}_1$ is not continuous. In this case, clause $(PA2)$ comes in handy as it serves the same role as $(PA2')$ but does not require closure under $\oplus$. It *does*, however, require that the space is closed under segmentation (Definition 3.10). The choice between $(PA2)$ and $(PA2')$ will not be important in this paper until Theorem 7.4. $\dashv$

Proposition 3.8 justifies the assumption of continuity in Definition 3.11. Here p corresponds to r. If we use Proposition 3.8 as a justification for the generalization, the reader may wonder why we did not additionally assume the continuity of the induced map into trajectories defined by

$$\bar{p}(\bar{u}, x)\colon [0, |\bar{u}|] \to X \qquad \bar{p}(\bar{u}, x)(t) = p(\bar{u}_{<t}, x) \qquad 0 \leqslant t \leqslant |\bar{u}|. \tag{4}$$

This is because continuity is implied by Corollary 3.15 below.

Lemma 3.13. *Let Z be a topological space and let $f\colon Z\times[0,T]\to\mathbb{R}$ and $h\colon Z\to[0,T]$ be continuous. Then, the function $g\colon Z\to\mathbb{R}$ defined by*

$$g(z)=\sup\{f(z,t)\mid t\in[0,h(z)]\}$$

is continuous.

Proof. Let $a<b$ be real numbers. We will show that $g^{-1}(a,b)$ is open in Z. Let

$$\begin{aligned}E&=\{z\mid\exists t\in[0,T](t\leqslant h(z)\wedge f(z,t)>a)\},\ \text{and}\\ A&=\{z\mid\forall t\in[0,T](t\leqslant h(z)\to f(z,t)<b)\}.\end{aligned}$$

By the continuity of h and f, we can replace "$\leqslant$" by "$<$" in the definition of E, so

$$E=\{z\mid\exists t\in[0,T](t<h(z)\wedge f(z,t)>a)\}.$$

We now have $g^{-1}(a,b)=E\cap A$; thus, it suffices to show that E and A are open. The set E is the projection of $\{(z,t)\mid t<h(z)\}\cap f^{-1}(a,\infty)$ which is open by the continuity of h and f. Thus, it remains to show that A is open. Let $z_0\in A$. We will find an open neighborhood O of z_0 with $O\subset A$. Let $O^b=f^{-1}(-\infty,b)$. Since it is open by the continuity of f, for each $t\in[0,h(z_0)]$ we have $(z_0,t)\in O^b$ and it has a rectangular open neighbourhood $O^t\times I^t$ in O^b:

$$(z_0,t)\in O^t\times I^t\subset O^b. \tag{5}$$

By compactness, find $t_0,\dots,t_n$ such that $I^{t_0},\dots,I^{t_n}$ cover $[0,h(z_0)]$. Let $O=O^{t_0}\cap\cdots\cap O^{t_n}$. Clearly, $z_0\in O$. It remains to show that $O\subset A$. So let $z\in O$ and $t\in[0,h(z_0)]$ be arbitrary. Let k be such that $t\in I^{t_k}$. However, then $(z,t)\in O^{t_k}\times I^{t_k}\subset O^b$ by (5); thus, $f(z,t)<b$ by the definition of O^b. □

Below, let X^* be the set of all functions $[0,T]\to X$, $T\in\mathbb{R}_{\geqslant 0}$.

Proposition 3.14. *Let $\mathcal{U}$ be as in Definition 3.10, Z any topological space, (X,d_X) a Polish metric space, and $\mathcal{X}$ as in (1) with metric as in (2). Suppose $p\colon\mathcal{U}\times Z\to X$ is continuous and define the function $\bar{p}\colon\mathcal{U}\times Z\to X^*$ as in (4), by*

$$\bar{p}(\bar{u},z)\colon[0,|\bar{u}|]\to X\qquad\bar{p}(\bar{u},x)(t)=p(\bar{u}_{<t},x)\qquad 0\leqslant t\leqslant|\bar{u}|. \tag{6}$$

Then, the range of $\bar{p}$ is a subset of $\mathcal{X}$, and $\bar{p}$ is continuous.

Proof. To check that $\bar{x}=\bar{p}(\bar{u},x)$ belongs to $\mathcal{X}$, simply note that by the continuity of p and of $\bar{u}\mapsto\bar{u}_{<t}$, $\bar{x}$ is also continuous. For the continuity of $\bar{p}$, it is enough to show that for all $(\bar{u}_0,z_0)\in\mathcal{U}\times Z$ and all ε, the inverse image of $B_{\mathcal{X}}(\bar{p}(\bar{u}_0,z_0),\varepsilon)$ is open. Thus, fix $(\bar{u}_0,z_0)\in\mathcal{U}\times Z$. Let $T=|u_0|$. Define $f\colon\mathcal{U}\times Z\times[0,T]\to\mathbb{R}$ by

$$f(\bar{u},z,t)=d_X(p(\bar{u}_{<t},z),p((\bar{u}_0)_{<t},z_0))+|T-|\bar{u}||.$$

By the continuity of $(\bar{u},t)\mapsto\bar{u}_{<t}$, of $\bar{u}\mapsto|\bar{u}|$ (see Definition 3.10), of p, and of the metric d_X, f is continuous. Let $h\colon\mathcal{U}\times Z\to[0,T]$ be defined by $h(\bar{u},z)=|\bar{u}|$ which

is again continuous. By Lemma 3.13, the function $g\colon \mathcal{U} \times Z \to \mathbb{R}$ given by $g(\bar{u}, z) = \sup\{f(\bar{u}, z, t) \mid t \in [0, h(\bar{u}, z)]\}$ is continuous. But $g(\bar{u}, z) = \varrho_{\mathcal{X}}(\bar{p}(\bar{u}, z), \bar{p}(\bar{u}_0, z_0))$. Now consider the inverse image

$$\begin{aligned} \bar{p}^{-1} B_{\mathcal{X}}(\bar{p}(\bar{u}_0, z_0), \varepsilon) &= \{(\bar{u}, z) \in \mathcal{U} \times Z \mid g(\bar{u}, z) < \varepsilon\} \\ &= g^{-1}(-\varepsilon, \varepsilon). \end{aligned}$$

which is open by the continuity of g. □

Corollary 3.15. *Suppose $p\colon \mathcal{U} \times X \to X$ is a path action. Let $\bar{p}$ be defined as in (4). Then, $\bar{p}\colon \mathcal{U} \times X \to \mathcal{X}$ is continuous.*

Proof. Choose $Z = X$ in Proposition 3.14. □

For our purposes a full-fledged action as described in Definition 3.11 is often not necessary. In our setup (Definition 4.1) the environment will always have a unique initial state $x_0 \in X$ for the robot; thus, all trajectories will start from that point. It is only a technicality, as switching the initial state can be formalized as switching the environment from (X, x_0) to (X, x_1). However, it will make mathematics easier for our considerations of covering spaces. Because of this, it will often be enough to consider the restriction $p \restriction (\mathcal{U} \times \{x_0\})$. Thus, we define:

Definition 3.16. Let (X, x_0) be a pointed space. An *initialized path action* is a continuous $p\colon \mathcal{U} \to X$ such that $p(\varnothing) = x_0$. We also denote $p\colon \mathcal{U} \to (X, x_0)$, or even $p\colon (\mathcal{U}, \varnothing) \to (X, x_0)$, to emphasize that it is a map between pointed spaces. As for path actions (4), given an initialized path action p, define $\bar{p}\colon \mathcal{U} \to \mathcal{X}$ by

$$\bar{p}(\bar{u})(t) = p(\bar{u}_{<t}), \quad t \leqslant |\bar{u}|. \tag{7}$$

⊣

Remark 3.17. Given a path action $p\colon \mathcal{U} \times X \to X$, the function $p_{x_0}\colon \mathcal{U} \to X$ defined by $p_{x_0}(\bar{u}) = p(\bar{u}, x_0)$ is an initialized path action. ⊣

Proposition 3.18. *Let $p\colon \mathcal{U} \to (X, x_0)$ be an initialized path action. Then, $\bar{p}$ is continuous.*

Proof. Choose $Z = \{x_0\}$ in Proposition 3.14, and identify $\mathcal{U} \times \{x_0\}$ with $\mathcal{U}$. □

4 History Information Spaces

In this section we first define environments and how trajectories in X become also trajectories in the sensory space (Section 4.1). Then we generalize history information spaces in Section 4.2.

4.1 Trajectories in environments

We want to now make precise the idea that two environments are indistinguishable from the point of view of some mobile robot. We introduce some more definitions. Fix a Polish space of control signals $\mathcal{U}$ as in Definition 3.10. Let the *observation space* Y be any topological space, corresponding to the set of all outputs of a given sensor connected to the robot. When comparing environments, they all should share $\mathcal{U}$ and Y because these are the "interfaces" between the robot and its environment. If they are different, it means that the robot has different actuators or different sensors, and comparing such situations is beyond our present analysis. Thus, we consider that $\mathcal{U}$ and Y are fixed for the rest of the paper.

The algebra of Borel sets in a Polish space is the smallest σ-algebra containing the basic open sets. A function from a Polish space to another is *Borel* if the inverse image of every open set is Borel. Equivalently, the inverse image of every Borel set is Borel.

Definition 4.1. An *environment* is a tuple $E = (X, x_0, h, p)$ where X is a Polish space, $x_0 \in X$ is the *initial position*, $h \colon X \to Y$ a Borel *sensor mapping* [21], and p is either an initialized path action $p \colon \mathcal{U} \to X$ or a path action $p \colon \mathcal{U} \times X \to X$ (Definitions 3.11 and 3.16). The pair (h, p) is called a *sensorimotor structure* on (X, x_0). We will assume that p is an initialized path action unless mentioned otherwise. $\dashv$

We require h to be Borel because continuity is too strong of a requirement in general, but to have no requirements at all would make working with h difficult. The class of Borel functions is loose enough to include all functions that are generally interesting in this context, such as piecewise continuous functions. Unlike measurable sets, Borel sets are topologically invariant (preserved by homeomorphisms). If we were to define a Radon measure on X, then all Borel functions would automatically be measurable. Moreover, all Borel functions are continuous on a co-meager set. (Co-meager sets are sets containing an intersection of countably many dense open sets.) Yet another benefit for us is that the composition of Borel functions is a Borel function and hence measurable.

Let $\mathcal{Y}$ be the space of measurable functions $\bar{y} \colon [0, T) \to Y$, just as $\mathcal{U}$ is the set of measurable functions into U (Definition 3.1). As defined in (7), each path $\bar{u} \colon [0, T) \to U$ in $\mathcal{U}$ generates a path $\bar{p}(\bar{u}) = \bar{x}_{\bar{u}} \colon [0, T] \to X$ defined by

$$\bar{x}_{\bar{u}}(t) = p(\bar{u}_{<t}) = \bar{p}(\bar{u})(t) \tag{8}$$

for all $0 \leqslant t < T$. This $\bar{x}_{\bar{u}}$ is the trajectory that the robot will traverse in the configuration space X starting from its initial position x_0 and applying the control given by $\bar{u}$. This trajectory generates a unique path $\bar{y}_{\bar{u}} \colon \mathbb{R}_{\geqslant 0} \to Y$ in the robot's observation space defined by

$$\bar{y}_{\bar{u}}(t) = h(\bar{x}_{\bar{u}}(t)) = h(\bar{p}(\bar{u})(t)) = h(p(\bar{u}_{<t})). \tag{9}$$

Since $\bar{x}_{\bar{u}}$ is continuous and h is Borel, $\bar{y}_{\bar{u}}$ is measurable.

Define the induced sensory trajectory map $\bar{h} \colon \mathcal{X} \to \mathcal{Y}$ by

$$\bar{h}(\bar{x})(t) = h(\bar{x}(t)). \tag{10}$$

Using this notation, we have

$$y_{\bar{u}} = \bar{h}(\bar{p}(\bar{u})). \tag{11}$$

4.2 Continuous-time history information states

The following definition can be thought of as a continuous-time version of the history information space $\mathcal{I}$ [21, Ch. 11], and [23, 31, 34].

Definition 4.2 (History information space)**.** A pair $\eta = (\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$ is a *history information state*. The set of all information states $\mathcal{U} \times \mathcal{Y}$ is denoted by $\mathcal{I}$. $\dashv$

Now, each environment $E = (X, x_0, h, p)$ carves out a subspace of $\mathcal{I}$ which contains only those histories that are possible in E.

Definition 4.3. Given an environment E, let $\mathcal{I}^E$ be the set of all pairs $(\bar{u}, \bar{y})$ where $\bar{y} = \bar{y}_{\bar{u}}$. To specify in which environment $\bar{y}$ was obtained from $\bar{u}$, we denote $\bar{y}^E_{\bar{u}} = \bar{y}_{\bar{u}}$. $\dashv$

5 Indistinguishability of Environments

In this section we deal with the equivalence relations of indistinguishability of environments. We will later introduce the non-symmetric relation of *strong* indistinguishability in Section 6.3.

5.1 Main construction

We will now present the main formal ingredients of the theory of indistinguishability.

Definition 5.1. Two environments $E = (X, x_0, h, p)$ and $E' = (X', x'_0, h', p')$ are *$\mathcal{I}$-equivalent*, if for all $\bar{u} \in \mathcal{U}$, $\bar{y}^E_{\bar{u}} = \bar{y}^{E'}_{\bar{u}}$. Denote the $\mathcal{I}$-equivalence relation by $\equiv^{\mathcal{I}}$. $\dashv$

We have the following:

Lemma 5.2. *The following are equivalent:*

1. $E \equiv^{\mathcal{I}} E'$,
2. $\mathcal{I}^E = \mathcal{I}^{E'}$,
3. $\bar{h} \circ \bar{p} = \bar{h}' \circ \bar{p}'$,
4. $h \circ p = h' \circ p'$.

Proof. **1.⇒2.** follows from the definitions of $\equiv^{\mathcal{I}}$ and $\mathcal{I}^E$, $\mathcal{I}^{E'}$.

2.⇒3. Suppose $\bar{u} \in \mathcal{U}$. Then, there is only one $\bar{y} \in \mathcal{Y}$ such that the pair $(\bar{u}, \bar{y})$ is in $\mathcal{I}^E$ and it is $\bar{y} = \bar{y}^E_{\bar{u}}$ which by (11) equals

$$\bar{y}^E_{\bar{u}} = \bar{h}(\bar{p}(\bar{u})). \tag{12}$$

However, since $\mathcal{I}^E = \mathcal{I}^{E'}$, also $(\bar{u}, \bar{y}^E_{\bar{u}}) \in \mathcal{I}^{E'}$. By the fact that in $\mathcal{I}^{E'}$ there is only one pair with the first coordinate equal to $\bar{u}$, we have $\bar{y}^E_{\bar{u}} = \bar{y}^{E'}_{\bar{u}}$ and using (11) again we have $\bar{y}^{E'}_{\bar{u}} = \bar{h}'(\bar{p}'(\bar{u}))$ and using (12) we have $\bar{h}(\bar{p}(\bar{u})) = \bar{h}'(\bar{p}'(\bar{u}))$. By the arbitrary choice of $\bar{u}$, we conclude 3.

3.⇒4. Let $\bar{u} \in \mathcal{U}$. By (7) we can write

$$\bar{p}(\bar{u}')(|\bar{u}|) = p(\bar{u}) \quad \text{and} \quad \bar{p}'(\bar{u}')(|\bar{u}|) = p'(\bar{u}). \tag{13}$$

Using (10), we now obtain:

$$\begin{aligned} h(p(\bar{u})) &\overset{(13)}{=} h(\bar{p}(\bar{u}')(|\bar{u}|)) \overset{(10)}{=} \bar{h}(\bar{p}(\bar{u}'))(|\bar{u}|) = (\bar{h} \circ \bar{p})(\bar{u}')(|\bar{u}|) \\ &\overset{3.}{=} (\bar{h}' \circ \bar{p}')(\bar{u}')(|\bar{u}|) = \bar{h}'(\bar{p}'(\bar{u}'))(|\bar{u}|) \overset{(10)}{=} h'(\bar{p}'(\bar{u}')(|\bar{u}|)) \\ &\overset{(13)}{=} h'(p'(\bar{u})). \end{aligned}$$

4.⇒1. Let $\bar{u} \in \mathcal{U}$. We need to show that $\bar{y}^E_{\bar{u}} = \bar{y}^{E'}_{\bar{u}}$. By (9), for all $t < |\bar{y}_{\bar{u}}|$ we have

$$\bar{y}^E_{\bar{u}} = h(p(\bar{u}_{<t})) \overset{4.}{=} h(p(\bar{u}_{<t})) = \bar{y}^{E'}_{\bar{u}},$$

which completes the proof.

□

Corollary 5.3. $\equiv^{\mathcal{I}}$ *is an equivalence relation.*

Proof. Follows easily from any of the characterizations given by Lemma 5.2. □

Thus, $\mathcal{I}$-equivalence means that no matter what the robot *does*, it cannot receive different sensory readings in these two environments. We return to the example of a circle and a line.

Example 5.4. Fix $U = \{-1, 1\}$ and some Polish observation space Y. Suppose $X = S^1$ is given as $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$ with the initial point $x_0 = e^0$. Suppose $h \colon S^1 \to Y$ is a continuous sensor mapping and given $\bar{u} \in \mathcal{U}$, the robot's state is given by $p(\bar{u}) = e^{i\theta(\bar{u})}$, where

$$\theta(\bar{u}) = \int_0^{|\bar{u}|} \bar{u}(t)dt.$$

This defines an environment $E = (X, x_0, h, p)$.

Now let $X' = \mathbb{R}$, $x'_0 = 0$, and $p'(\bar{u}) = \int_0^{|\bar{u}|} \bar{u}(t)dt$. To define $h' \colon \mathbb{R} \to Y$, let $f \colon \mathbb{R} \to S^1$ be the covering map $t \mapsto e^{it}$, and let $h' = h \circ f$. Now p' is a lifting of p, $f \circ p' = p$, and the following diagram commutes:

$$\begin{tikzcd} & (X', x'_0) \arrow[d, "f"] \arrow[dr, "h'"] & \\ (\mathcal{U}, \varnothing) \arrow[ur, "p'"] \arrow[r, "p"] & (X, x_0) \arrow[r, "h"] & Y. \end{tikzcd} \tag{14}$$

This diagram shows *pointed spaces* which are pairs (Z, z) with Z a space and $z \in Z$ a point. The arrows correspond to maps which take the selected point to the selected point. The selected point in $\varnothing \in \mathcal{U}$ is the empty control signal corresponding to $T = 0$. We have not shown the selected point in Y because our maps do not have a requirement to map x_0 or x'_0 to any particular point, although due to commutation, we know that $h'(x'_0) = h(x_0)$. From the above we have

$$h \circ p = h \circ (f \circ p') = (h \circ f) \circ p' = h' \circ p'.$$

Thus, by Lemma 5.2(1. $\Leftrightarrow$ 4), we have $E' \equiv^{\mathcal{I}} E$, and so the environments are indistinguishable.

Note that we *defined* h' from h using f. In Section 6.2 we will also see that such p' as above can always be obtained from any initialized path action p. This means that *no matter which* sensor mapping there is on the circle, the possibility that it is actually the line can never be ruled out. We will see in Section 6.2 (especially Theorem 6.5) that the notion of a covering space plays a key role here. $\dashv$

We can also formulate an equivalence in terms of "eternal" or unbounded trajectories. This will have the advantage of increased generality. We define:

Definition 5.5. A *branch* through $\mathcal{U}$ is defined to be a function $\mathbf{u} \colon \mathbb{R}_{\geqslant 0} \to U$ such that for all $t \in \mathbb{R}_{\geqslant 0}$, we have that $\mathbf{u} \restriction [0, t) \in \mathcal{U}$. Let $B\mathcal{U}$ be the set of branches through $\mathcal{U}$. Similarly, denote by $\mathbf{x}$ a branch through $\mathcal{X}$, i.e., a function $\mathbf{x} \colon \mathbb{R}_{\geqslant 0} \to X$ such that $\mathbf{x} \restriction [0, T] \in \mathcal{X}$ for all $T \in \mathbb{R}_{\geqslant 0}$. Denote the set of branches through $\mathcal{X}$ by $B\mathcal{X}$. Similarly let $B\mathcal{Y}$ be the set of branches through $\mathcal{Y}$ defined analogously. $\dashv$

Denote by $\mathbf{p}$ and $\mathbf{h}$ the natural extensions of $\bar{p}$ and $\bar{h}$ to the sets of branches:

$$\mathbf{p} \colon B\mathcal{U} \to B\mathcal{X}, \qquad \mathbf{p}(\mathbf{u})(t) = p(\mathbf{u}_{<t}), \tag{15}$$

$$\mathbf{h} \colon B\mathcal{X} \to B\mathcal{Y}, \qquad \mathbf{h}(\mathbf{x})(t) = h(\mathbf{x}(t)), \tag{16}$$

in which $\mathbf{u}_{<t} = \mathbf{u} \restriction [0, t)$ is defined just like for elements of $\mathcal{U}$.

For the bounded-domain paths, given a path action $p \colon \mathcal{U} \times X \to X$, each branch $\mathbf{u}$ through $\mathcal{U}$ generates a unique trajectory $\mathbf{x_u} = \mathbf{p}(\mathbf{u}) \colon \mathbb{R}_{\geqslant 0} \to X$ through the state space given by

$$\mathbf{x_u}(t) = p(\mathbf{u}_{<t}). \tag{17}$$

This trajectory generates a unique path $\mathbf{y_u} = \mathbf{h}(\mathbf{x}) \colon \mathbb{R}_{\geqslant 0} \to Y$ in the observation space defined by $\mathbf{h}(\mathbf{x})(t) = h(p(\mathbf{u} \restriction t))$. We write $\mathbf{y}^E_{\mathbf{u}}$ to specify the environment E in which it was computed.

Definition 5.6. The branches $\mathbf{x} \in B\mathcal{X}$ are called *full trajectories* and $\mathbf{y} \in B\mathcal{Y}$ are called the *full sensory histories*. The set of pairs $(\mathbf{u}, \mathbf{y}) \in B\mathcal{U} \times B\mathcal{Y}$ is the set of *full information histories* and we denote it by $\boldsymbol{\mathcal{I}}$. Analogously to Definition 4.3 we also define $\boldsymbol{\mathcal{I}}^E \subset \boldsymbol{\mathcal{I}}$ to be the set of the pairs of the form $(\mathbf{u}, \mathbf{y}^E_{\mathbf{u}})$ for a given environment E. $\dashv$

We can now reformulate our definition of equivalence in terms of full information histories. This definition is more general than the definition of $\mathcal{I}$-equivalence (Definition 5.1) in the sense that it includes it as a special case (see Section 5.3), but also gives the possibility of defining a whole class of filter-based equivalence relations. A *filter* is an equivalence relation F either on Y, on $\mathcal{Y}$, or on $B\mathcal{Y}$ usually so that an equivalence on Y induces equivalences on the function spaces through pointwise application. Examples of filters are the gap-sensor (Example 8.1), the beam-sensor (Example 8.6) or considering histories up to homeomorphisms (see Section 5.2).

Definition 5.7. Given an equivalence relation F on the set of full information histories $B\mathcal{U} \times B\mathcal{Y}$, let $\equiv_F$ be an equivalence relation on the set of all environments E, E' such that

$$E \equiv_F E'$$

if and only if for all $\mathbf{u} \in B\mathcal{U}$, $((\mathbf{u}, \mathbf{y}^E_{\mathbf{u}}), (\mathbf{u}, \mathbf{y}^{E'}_{\mathbf{u}})) \in F$. We call this *filter based full historical equivalence* or just *filter based equivalence*. If F is the identity relation, we drop it from the notation, so $\equiv$ is the same as $\equiv_{\mathrm{id}}$. We call this *full historical equivalence* or just *historical equivalence*. $\dashv$

Similarly as for $\equiv^{\mathcal{I}}$, we have:

Lemma 5.8. *The following are equivalent:*

1. $E \equiv E'$,
2. $\boldsymbol{\mathcal{I}}^E = \boldsymbol{\mathcal{I}}^{E'}$,
3. $\mathbf{h} \circ \mathbf{p} = \mathbf{h}' \circ \mathbf{p}'$,
4. $\bar{h} \circ \bar{p} = \bar{h}' \circ \bar{p}'$,
5. $h \circ p = h' \circ p'$,

Proof. The implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. are proved similarly as the same ones in Lemma 5.2, and the implications 4. $\Rightarrow$ 5. $\Rightarrow$ 1. similarly as 3. $\Rightarrow$ 4. $\Rightarrow$ 1. in Lemma 5.2. The only one requiring a new argument is 3. $\Rightarrow$ 4. which we do now.

3.$\Rightarrow$4. Suppose $\bar{u} \in \mathcal{U}$. Then, by the property of being extensive (Definition 3.10), we can find $\mathbf{u} \in B\mathcal{U}$ with $\mathbf{u}_{<|\bar{u}|=\bar{u}}$. Let $t \leqslant |\bar{u}|$. Then,

$$\begin{aligned}
\bar{h}(\bar{p}(\bar{u}))(t) &\overset{(10)}{=} h(\bar{p}(\bar{u})(t)) \overset{(7)}{=} h(p(\bar{u}_{<t})) = h(p((\mathbf{u}_{<|\bar{u}|})_{<t})) \\
&\overset{t\leqslant|\bar{u}|}{=} h(p(\mathbf{u}_{<t})) \overset{(15)}{=} h(p(\mathbf{u})(t)) \overset{(16)}{=} \mathbf{h} \circ \mathbf{p}(t) \\
&\overset{3.}{=} \mathbf{h}' \circ \mathbf{p}'(t) \overset{(16)}{=} h'(p'(\mathbf{u})(t)) \overset{(15)}{=} h'(p'(\mathbf{u}_{<t})) \\
&= h'(p'((\mathbf{u}_{<|\bar{u}|})_{<t})) = h'(p'(\bar{u}_{<t})) \overset{(10)}{=} \bar{h}'(\bar{p}'(\bar{u}))(t).
\end{aligned}$$
$\square$

5.2 Examples of filter-based equivalence relations

Example 5.9. Define $(\mathbf{u}, \mathbf{y}^E_\mathbf{u}) \approx (\mathbf{u}, \mathbf{y}^{E'}_\mathbf{u})$, if and only if there is a homeomorphism $f\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $\mathbf{y} \circ f = \mathbf{y}'$ and $\mathbf{u} \circ f = \mathbf{u}'$. Then consider $\equiv_\approx$. Clearly, $\equiv$-equivalence implies $\equiv_\approx$-equivalence by choosing the homeomorphism to be the identity. The relation $\equiv_\approx$ pertains to robots whose time perception is only relational: there is memory and knowledge of which chronological order the events occurred in, but not of how much time passed between them. This is because the set of homeomorphisms $\mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is exactly the set of strictly order preserving bijections. $\dashv$

Example 5.10. Suppose F is an equivalence relation on Y making some observations indistinguishable. It induces an equivalence relation $\bar{F}$ on $\mathcal{Y} \cup B\mathcal{Y}$ through pointwise application: $(\bar{y}_0, \bar{y}_1) \in \bar{F}$ if $\mathrm{dom}(\bar{y}_0) = \mathrm{dom}(\bar{y}_1)$ and for all $t \in \mathrm{dom}(\bar{y}_0)$ we have $(\bar{y}_0(t), \bar{y}_1(t)) \in F$. Then, $\equiv_{\bar{F}}$ is an equivalence relation which equates those environments which cannot be distinguished by any robot that is equipped with this filter F. We will abuse the notation and denote $\equiv_F$ in this case instead of $\equiv_{\bar{F}}$ and call it *filter based equivalence induced by F.* $\dashv$

Example 5.11. Suppose the robot has a high-level sensor which reports the result of a low-level sensorimotor interaction. For example repeatedly pushing against a wall can provide the data of how hard the wall is and moving whiskers can generate data about the presence of obstacles or texture of surfaces [5]. This can be expressed by defining a labeling on the set of information histories of fixed length $g\colon \{(\bar{u}, \bar{y}) \in \mathcal{I} : |\bar{y}| = 1\} \to L$, where L is some set of labels. This induces an equivalence relation on $B\mathcal{Y}$ as follows:

$$(\mathbf{u}, \mathbf{y}) \in F_g \iff \forall n \in \mathbb{N}\Big(g(\mathbf{u} \restriction [n, n+1)) = g(\mathbf{y} \restriction [n, n+1))\Big),$$

or as a moving window as follows:

$$(\mathbf{u}, \mathbf{y}) \in F'_g \iff \forall t \in \mathbb{R}_{\geqslant 0}\Big(g(\mathbf{u} \restriction [t, t+1)) = g(\mathbf{y} \restriction [t, t+1))\Big).$$

Then, $\equiv_{F_g}$ or $\equiv_{F'_g}$ capture the equivalence with respect to this filtering. $\dashv$

Open Problem 5.12. *How to generalize the notion of* derived information spaces *[21, 31] and of* sufficient equivalence relations *[34] to the continuous framework?*

5.3 Equivalence of equivalences

Both equivalence relations $\equiv^\mathcal{I}$ and $\equiv$ are defined in a natural way of what would it mean for two environments to be indistinguishable. In fact, they turn out to be the same:

Theorem 5.13. *For all environments E and E' we have $E \equiv^\mathcal{I} E'$ iff $E \equiv E'$.*

Proof. Apply Lemma 5.2(4) and Lemma 5.8(5). $\square$

In view of Theorem 5.13 it is enough to only talk about equivalence relations of the form $\equiv_F$ for some equivalence relation F either on Y, on $\mathcal{Y}$ or on $B\mathcal{Y}$ (see Examples 5.11 and 5.10). This opens an intriguing avenue for future research, but in this paper we only focus on $\equiv$ with F being the identity.

We have now defined a class of equivalence relations on tuples (X, x_0, h, p) which are, for all intents and purposes, continuous time dynamical systems with a robotic twist (the sensor mapping h and the nature of the action p). The equivalence relations defined are motivated by robotics because they are based on comparing orbits (robot's trajectories) in terms of the sensor mapping h. We now state another open problem:

Open Problem 5.14. *What is the complexity of various $\equiv_F$'s in terms of descriptive complexity and Borel reducibility [11]? How do they compare to other known equivalence relations on dynamical systems?*

6 Homomorphisms and Covering Spaces

Upon seeing an equivalence relation, it is natural to ask *which maps witness it*? For example, a homeomorphism witnesses homotopy equivalence but does not witness isometric equivalence.

6.1 Homomorphisms of environments

Definition 6.1. Given environments $E = (X, x_0, h, p)$ and $E' = (X', x'_0, h', p')$, a map $f\colon X' \to X$ is a *homomorphism* (also denoted as $f\colon E' \to E$), if the following conditions are satisfied:

(HOM1) $f(x'_0) = x_0$,

(HOM2) for all $\bar{u} \in \mathcal{U}$, $f(p'(\bar{u})) = p(\bar{u})$,

(HOM3) for all $x \in X$, $h'(x) = h(f(x))$.

This is to say that f is such that the following diagram, which is already familiar from Example 5.4(14), commutes:

$$\begin{tikzcd} & (X', x'_0) \arrow[d, dashed, "f"] \arrow[dr, "h'"] & \\ (\mathcal{U}, \varnothing) \arrow[ur, "p'"] \arrow[r, "p"] & (X, x_0) \arrow[r, "h"] & Y. \end{tikzcd}$$

If p is a initialized path action (see Definition 4.1), then we require a strengthening of (HOM2):

(HOM2)' For all $\bar{u} \in \mathcal{U}$ and all $x' \in X'$, $f(p'(\bar{u}, x')) = p(\bar{u}, f(x'))$.

Recall that we are using pointed spaces. The arrows correspond to maps which take the selected point to the selected point. The selected point in $\mathcal{U}$ is the empty control signal corresponding to $T = 0$. We have not shown the selected point in Y because the sensor mappings h and h' do not have a requirement to map x_0 or x'_0 to any particular point, although due to commutation of the diagram (if holds), we know that $h'(x'_0) = h(x_0)$. $\dashv$

It is straightforward to see that the existence of a homomorphism is a sufficient condition for the equivalence to take place:

Theorem 6.2. *If there is a homomorphism $f\colon E' \to E$, then $E \equiv E'$ and $E \equiv^{\mathcal{I}} E'$.*

Proof. Consider the commutative diagram of Definition 6.1. Since the triangle on the left commutes, we have $f \circ p' = p$, and from the triangle on the right, we have $h' = h \circ f$. So,

$$h \circ p = h \circ (f \circ p') = (h \circ f) \circ p' = h' \circ p'.$$

Applying Lemmas 5.2 and 5.8 we have the result. $\square$

6.2 Covering maps of environments

Covering maps have the role of "unravelling" fundamental groups. A closed loop is the image of a non-closed path in the covering space. This is why they are natural to analyse loop closure. The topologist will also readily recognize the idea of covering spaces from the commutative diagrams in (14) and Definition 6.1. Before we can exploit this idea, we prove the following:

Lemma 6.3. *The space $\mathcal{U}$ is contractible.*

Proof. We will prove a slightly stronger property that $\{\varnothing\} \subset \mathcal{U}$ is a strong deformation retract of $\mathcal{U}$. Define $F\colon \mathcal{U} \times [0,1] \to \mathcal{U}$ by $F(\bar{u}, t) = \bar{u}_{<\theta(t)}$ where $\theta(t) = -\ln(t)$ and by convention $-\ln(0) = \infty$ and $\bar{u}_{<\infty} = \bar{u}$. By Definition 3.10, F is continuous because the map $(\bar{u}, t) \mapsto \bar{u}_{<t}$ is. Clearly, $F(\bar{u}, 0)$ is the identity, $F(\varnothing, t) = \varnothing$ for all t, and $F(\bar{u}, 1) = \bar{u}_{<0} = \varnothing$ for all $\bar{u}$. Thus, F is a strong deformation retraction to $\{\varnothing\}$. $\square$

We say that $A \subset X$ is *reachable*, if for all $a \in A$ there is $\bar{u} \in \mathcal{U}$ with $p(\bar{u}) = a$. We say that the environment is *fully reachable*, if X is reachable. A *covering map* $f\colon (X', x'_0) \to (X, x_0)$ is an onto map which is a local homeomorphism. A pointed space (X', x') is a *covering space* of (X, x), if there is a *covering map* from (X', x') to (X, x).

Lemma 6.4. *Assume that $\mathcal{U}$ is path-connected and locally path-connected. $E = (X, x_0, h, p)$ is a fully reachable environment, and (X', x'_0) is a covering space of (X, x_0) witnessed by the covering map f. Then, there is unique initialized path action $p'\colon \mathcal{U} \to X'$ such that $f \circ p' = p$ and a unique sensing mapping $h'\colon X' \to Y$ such that $h' = h \circ f$.*

Proof. Since $\mathcal{U}$ is simply connected (by Lemma 6.3), there is a unique lift of $p\colon (\mathcal{U}, \varnothing) \to (X, x_0)$ to $p'\colon (\mathcal{U}, \varnothing) \to (X', x_0')$; see Propositions 1.33 and 1.34 in [16]. This p' is a continuous map such that $f \circ p' = p$ and $p'(\varnothing) = x_0'$. Thus, it satisfies the conditions of Definition 3.16 and is an initialized path action. The uniqueness and existence of h' follow from its definition. $\square$

Theorem 6.5. *Assume that $\mathcal{U}$ is path-connected and locally path-connected, $E = (X, x_0, h, p)$ is a fully reachable environment, and (X', x_0') is a covering space of (X, x_0). Then, there is a sensorimotor structure (h', p') on (X', x_0') making it into environment $E' = (X', x_0', h', p')$ such that $E \equiv E'$.*

Proof. Let f witness that (X', x_0') is a covering space of (X, x_0) and let h' and p' be the functions given by Lemma 6.4. Then, they make f into homomorphism $E' \to E$, and the result follows from Theorem 6.2. $\square$

We assumed above that $\mathcal{U}$ is path-connected and locally path-connected. We show that these assumptions are satisfied for $\mathcal{U} = \mathcal{U}_M$ of Definition 3.1:

Proposition 6.6. *Let $\mathcal{U}_M$ be as in Definition 3.1 with metric from Definition 3.4. Then, $\mathcal{U}_M$ is path-connected and locally path-connected.*

Proof. Let $\bar{u}_0, \bar{u}_1 \in \mathcal{U}_M$. We will construct a path $\gamma\colon [0,1] \to \mathcal{U}_M$ such that $\gamma(k) = u_k$ for $k \in \{0,1\}$ and for all $s \in [0,1]$ we will have $\varrho_{\mathcal{U}_M}(\gamma(s), u_0) + \varrho_{\mathcal{U}_M}(\gamma(s), u_1) = d(u_0, u_1)$. From this it follows that every ball $B_{\mathcal{U}_M}(\bar{u}, r)$, $r \in \mathbb{R}_+$, in $\mathcal{U}_M$ is path connected which implies the statement to be proved. Suppose w.l.o.g. $T_1 = |\bar{u}_1| \geqslant |\bar{u}_0| = T_0$. For $s \in [0,1]$ let $\gamma(s)$ be a path with $|\gamma(s)| = T_0 + (T_1 - T_0)s$ such that for all $t \in [0, T_0 + (T_1 - T_0)s)$ we have

$$\gamma(s)(t) = \begin{cases} \bar{u}_1(t), & \text{if } t < T_0 s \text{ or } t > T_0 \\ \bar{u}_0(t), & \text{otherwise.} \end{cases}$$

Being piecewise measurable, γ is measurable; thus, $\gamma \in \mathcal{U}_M$. Using the fact that $0 \leqslant s \leqslant 1$ one can verify that $|T_0 + (T_1 - T_0)s - T_0| + |T_1 - (T_0 + (T_1 - T_0)s)| = |T_1 - T_0|$. Then

$$\begin{aligned}
&\varrho_{\mathcal{U}_M}(\gamma(s), u_0) + \varrho_{\mathcal{U}_M}(\gamma(s), u_1) \\
&= \int_0^{T_0} d_U(\gamma(s)(t), u_0(t))dt + \int_0^{T_0} d_U(\gamma(s)(t), u_1(t))dt + |T_1 - T_0| \\
&= \int_0^{sT_0} d_U(\gamma(s)(t), u_0(t))dt + \int_{sT_0}^{T_0} d_U(\gamma(s)(t), u_0(t))dt \\
&\qquad + \int_0^{sT_0} d_U(\gamma(s)(t), u_1(t))dt + \int_{sT_0}^{T_0} d_U(\gamma(s)(t), u_1(t))dt + |T_1 - T_0|
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{sT_0} d_U(u_1(t), u_0(t))dt + \int_{sT_0}^{T_0} d_U(u_0(t), u_0(t))dt \\
&\qquad + \int_0^{sT_0} d_U(u_1(t), u_1(t))dt + \int_{sT_0}^{T_0} d_U(u_0(t), u_1(t))dt + |T_1 - T_0| \\
&= \int_0^{sT_0} d_U(u_1(t), u_0(t))dt + \int_{sT_0}^{T_0} d_U(u_0(t), u_1(t))dt + |T_1 - T_0| \\
&= \int_0^{T_0} d_U(u_1(t), u_0(t))dt + |T_1 - T_0| \\
&= \varrho_{\mathcal{U}_M}(u_0, u_1),
\end{aligned}
$$

which was to be proven. □

Corollary 6.7. *Assume that $\mathcal{U} = \mathcal{U}_M$, $E = (X, x_0, h, p)$ an environment, and (X', x_0') a covering space of (X, x_0). Then, there is a sensorimotor structure (h', p') on (X', x_0') making it into environment $E' = (X', x_0', h', p')$ such that $E \equiv E'$.*

Proof. By Lemma 6.6, $\mathcal{U}_M$ satisfies the assumptions of Theorem 6.5. □

Open Problem 6.8. *What are the minimal topological conditions for $\mathcal{U}$ such that Theorem 6.5 holds and when are they satisfied?*

Theorem 6.9. *Suppose $E = (X, x_0, h, p)$ and $E' = (X', x_0', h', p')$ are environments and that one of the following holds:*

(A) There exists a common covering space $(\tilde{X}, \tilde{x})$ of both (X, x_0) and (X', x_0') witnessed by covering maps f and f' respectively such that the lifts of p and p' are identical and such that $h \circ f = h' \circ f$.

(B) There exists a space $(\hat{X}, \hat{x}_0)$ such that both (X, x_0) and (X', x_0') are covering spaces of $(\hat{X}, \hat{x}_0)$ witnessed by covering maps f and f' respectively such that $f \circ p = f' \circ p'$ and there is $\hat{h}\colon \hat{X} \to Y$ such that h and h' are lifts of $\hat{h}$ along the respective covering maps.

Then, $E \equiv E'$.

Proof. In the first case, denote by $\tilde{p}$ the lift of p (and p') and by $\tilde{h} = h \circ f = h' \circ f$. Then, it is clear that f and f' are homomorphisms from $\tilde{E} = (\tilde{X}, \tilde{x}_0, \tilde{h}, \tilde{p})$ to E and E' respectively. Thus, we have $\tilde{E} \equiv E$ and $\tilde{E} \equiv E'$. By transitivity we have $E \equiv E'$. Note that the fact that $\equiv$ is an equivalence relation follows from Corollary 5.3 and Theorem 5.13.

For the second case, denote $\hat{p} = f \circ p = f \circ p'$ and we have that f and f' are homomorphisms from E and E' respectively to $\hat{E} = (\hat{X}, \hat{x}_0, \hat{h}, \hat{p})$; then apply transitivity again. □

Example 6.10 (An application to topology)**.** We can use Corollary 6.9 to prove that certain spaces *do not* have a common covering space, if we can show that a robot can distinguish between them. For example consider a robot which can detect the local homeomorphism type of its environment (for more on this see Theorem 8.7 in Section 8). Then, this robot can easily detect a difference between the following two spaces on Figure 3. In the space on the left it is possible to go forward along a 1-dimensional path and repeatedly bump into a 4-crossing (by circling around the left-most loop). In the space on the right, however, no matter how the robot traverses along the 1-dimensional edges, it will sooner or later bump into a 3-crossing. U-turns midway are not made. Thus, $E \not\equiv E'$ and therefore by Corollary 6.9 there is no common covering space of both of them, neither there is a space which both of them are a covering space of. $\dashv$

6.3 Strong indistinguishability

Motivated by Theorem 6.5, we can define:

Definition 6.11. A pointed space (X, x_0) is *strongly indistinguishable* from the pointed space (X', x'_0), if for all sensorimotor structures (h, p) on (X, x_0) there is a sensorimotor structure (h', p') on (X', x'_0) such that $E \equiv E'$ where $E = (X, x_0, h, p)$ and $E' = (X', x'_0, h', p')$. $\dashv$

The relation of strong indistinguishability is not symmetric, but it is transitive and reflexive; thus, it determines a quasiorder on environments. The higher an environment is in this ordering the more ambiguities it has. At the top are the universal covering spaces and at the bottom are the quotients with respect the equivalence relation $h(x) = h(x')$ (see [34] for related concepts). See Figure 2 and it's caption.

7 Equivalence Characterization and Bisimulation

Theorem 6.9 gives sufficient conditions to decide when two spaces are $\equiv$-equivalent via covering spaces and covering maps. It is also appealing from the point of robotics because a visual sensor will typically report information about the geometric structure of the environment and covering spaces have the property of preserving the local topological structure. The covering maps, however, do note characterize the equivalence. For example, it is not hard to come up with examples where a projection map is in the role of a homomorphism, or where no homomorphism exists.

Example 7.1. For example let $X = S^1$ and $X' = S^1 \times [0, 1]$. Let $h\colon X \to \mathbb{R}$ and $h'\colon X' \to R$ be defined so that for all $\theta \in [0, 2\pi)$ and all $t \in [0, 1]$,

$$h(e^{i\theta}) = h'(e^{i\theta}, t) = \sin(\theta).$$

Thus, h' does not depend on t. Further, let the initialized path actions be some functions $p\colon \mathcal{U} \to X$ and $p'\colon \mathcal{U} \to X'$ such that $\mathrm{pr}_1 \circ p' = p$ where pr_1 is the projection to the first coordinate. Now the projection mapping pr_1 is, in fact, a witness of the equivalence between these two environments, but of course there are no common covering spaces because these must be local homeomorphisms. $\dashv$

It is even possible to have two spaces which are equivalent but there is no map at all that witnesses that:

Example 7.2. Let $X = X' = \Delta \vee I$ where Δ is a 2-simplex and I is a 1-simplex, and $\vee$ means that they are glued at one point, for example at a 0-face of both. Thus, it looks like a kite. Suppose $h \colon X \to \{-1, 0, 1\}$ is such that $h(x) = -1$ for all x in the 2-simplex and $h(x) = 1$ for x in the 1-simplex, except $h(x) = 0$ at the point where they are glued. For h' the numbers are flipped so that $h' = -h$. Now if a continuous $f \colon X \to X'$ commutes with the sensor mappings, then it is not surjective. It is not hard to come up with initialized path actions, however, which make the environments both equivalent and fully reachable. See Example 8.6 for another pair of such environments. $\dashv$

As a solution we will use a notion of bisimulation in the continuous setting. There is a lot of literature on bisimulation in the topological and continuous setting, especially in the context of hybrid systems [3, 4, 10, 17].

Definition 7.3. Let $E = (X, x_0, h, p)$ and $E' = (X', x'_0, h', p')$ be environments where p and p' are path actions (not initialized ones, see Definition 4.1). A binary relation $R \subset X \times X'$ is a *bisimulation* between E and E', if $(x_0, x'_0) \in R$ and for all $(x, x') \in X \times X'$ the following holds: If $(x, x') \in R$, then $h(x) = h(x')$ and for all $\bar{u} \in \mathcal{U}$, also $(p(\bar{u}, x), p(\bar{u}, x')) \in R$. $\dashv$

Theorem 7.4. *Suppose $E = (X, x_0, h, p)$ and $E' = (X', x'_0, h', p')$ are environments where p and p' are path actions (not initialized ones, see Definition 4.1) and assume that $\mathcal{U}$ is closed under $\oplus$. Then, $E \equiv E'$ if and only if there is a bisimulation $R \subset X \times X'$.*

Proof. We work with $\equiv^{\mathcal{I}}$ instead of $\equiv$ as justified Suppose $E \equiv^{\mathcal{I}} E'$. Then, define $R = \{(p(\bar{u}, x_0), p'(\bar{u}, x'_0)) \mid \bar{u} \in \mathcal{U}\}$. By choosing $\bar{u} = \varnothing$, we have $(x_0, x'_0) \in R$. Suppose $(x, x') \in R$. Let $\bar{u} \in \mathcal{U}$ witness this. Then

$$h(x) = h(p_{x_0}(\bar{u})) = h'(p'_{x_0}(\bar{u})) = h'(x').$$

Here we used the notation from Remark 3.17 and Lemma 5.2(4.). Now let $\bar{u}_1$ be arbitrary. Then, by Definition 3.11(PA2) and Remark 3.3(4) we have

$$p(\bar{u}_1, x) = p(\bar{u}_1, p(\bar{u}, x_0)) = p(\bar{u} \oplus \bar{u}_1, x_0)$$

and

$$p'(\bar{u}_1, x) = p(\bar{u}_1, p'(\bar{u}, x_0)) = p'(\bar{u} \oplus \bar{u}_1, x_0).$$

Thus, $\bar{u} \oplus \bar{u}_1$ witnesses that $(p(\bar{u}_1, x), p(\bar{u}_1, x')) \in R$. This completes the proof of the "only if"-part.

Suppose now that R is a bisimulation relation on $X \times X'$. Then, by definition of bisimulation $h(x_0) = h(x'_0)$, and for all $\bar{u}$, also $h(p(\bar{u}, x_0)) = h'(p'(\bar{u}, x'_0))$. By Lemma 5.2, we are done. $\square$

Open Problem 7.5. *Bisimulation often arises in the context of modal logic and Kripke models [33] and has also been studied by the present authors in the context of robotics and minimal sufficient equivalence relations [34]. Can a closer connection to these areas established by Theorem 7.4 be made?*

8 Bringing It All Together

We now circle back to the main questions of interest, raised in Sections 1 and 2. An important special case of a sensor mapping h is one which reports invariant information about the topological or metric properties of the local neighbourhood of the agent. Call such sensor mapping *geometry based*. This is a typical function of distance measurements and visual sensors in general. Covering maps preserve local topological structure, and if additionally required to be local isometries, also local metric structure. Therefore covering maps naturally preserve geometry based sensor mappings. This enables applying our framework to a diverse number of cases in theoretical robotics and we synthesize it in Theorem 8.7 and Corollary 8.8 below. These results can be seen as a culmination of this paper and the original motivation to explore this topic.

Recall the setup of Section 3.2. Let $O(X)$ be the set of open subsets of a Polish space X and suppose that $\xi\colon X \to O(X)$ is some *neighborhood function*, meaning that for all $x \in X$, we have $x \in \xi(x)$. We will consider systems (X, x_0, h, p), where $h(x)$ is either a metric or a topological invariant of $\xi(x)$, meaning that $\xi(x) \sim \xi(x') \to h(x) = h(x')$, where $\sim$ is either isometry or homeomorphism. The idea is that $\xi(x)$ is a set visible from x, and h is some sensor mapping that loses unnecessary information. For example, we revisit the gap-navigation trees of [30] (recall Section 2.2).

Example 8.1. In the gap-navigation trees setup, X is a closed subset of $\mathbb{R}^2$, and

$$\xi(x) = \{x' \in X \mid [x, x'] \subset X\}$$

is the set of all points reachable by a line from the robot's position as depicted on Figure 4(left). This set is open in X, but not open in $\mathbb{R}^2$. In fact, we can see that $\xi(x)$ is homeomorphic to a set A that has the property

$$B^2(0,1) \subseteq A \subseteq \bar{B}^2(0,1),$$

meaning that it is the closed 2-disk with some parts of the boundary missing. The missing parts of the boundary correspond precisely to the gaps in the visual field, or the discontinuities in the distance function. Thus, the sensor mapping $h(x) = g(\xi(x))$, where g reports the circular order of these discontinuities is a topological invariant of $\xi(x)$. The paper [30] addresses environments that are not simply connected, which the authors handle by having the robot place distinguishable pebbles in the environment at selected locations. Now we can go a step further and utilize Theorem 6.9 to construct indistinguishable but non-homeomorphic environments for such a sensor, one can start with a region which is not simply connected, take its covering space such that the covering map preserves the star-convex neighbourhoods up to homeomorphism. Too much distortion and they can be distinguished. Consider Figure 7. We denote the spaces depicted in (a), (b), and (c) respectively by A, B and C. The environment B is a covering space of A with a covering map which preserves the homeomorphism type of the star convex neighborhoods, i.e., commutes with the ξ-mapping defined above. C is also a covering space of A, but the covering map does not commute

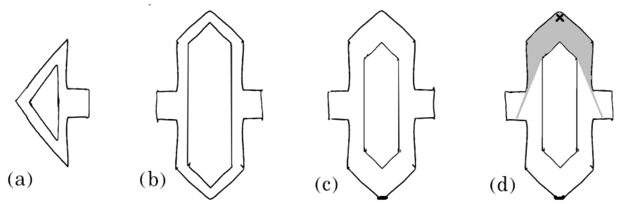

Figure 7: Environments (a) and (b) are indistinguishable by the gap-navigation sensor, but environments (a) and (c) as well as (b) and (c) are distinguishable. This is witnessed by the star convex neighborhood depicted in (d).

with $h = g \circ \xi$ and so is distinguishable from A, and therefore also from B. This is witnessed by the star convex neighborhood of the point in the top corner in Figure 7(d). This neighborhood has four discontinuities and such is nowhere to be found in A, B or C. According to Theorem 6.5 there exists some other sensor mapping h' on C which makes it indistinguishable from A. $\dashv$

Example 8.2. In the wall-following robot scenario of [18] (Figure 5), the robot can sense only very locally. The polygonal environments in [18] are *metrically locally uniform* in the following sense:

Local metric uniformity. X is equipped with a metric d_X and for all $x \in X$ there is $\varepsilon_x > 0$ such that for all $0 < \delta < \varepsilon_x$ the subspaces $\bar{B}_X(x, \varepsilon_x)$ and $\bar{B}_X(x, \delta)$ are isometric. We call the isometry type of $\bar{B}_X(x, \varepsilon_x)$ the *local isometry type at* x.

Thus, let $\xi(x) = B_X(x, \varepsilon_x)$. Then, ξ encodes local metric structure. The sensor of [18] is now a metric invariant of ξ. Thus, in view of Theorem 6.9, if one wanted to construct environments indistinguishable by such sensor, one could start with covering spaces whose covering maps are local isometries. $\dashv$

Example 8.3. The problem of detecting graph isomorphism by exploring it [20, 27] is a problem of reconstructing a global map from local information. Graphs, viewed as 1-complexes, are *topologically locally uniform* in the following sense:

Local topological uniformity. X has a basis B such that for all $x \in X$ there is $O_x \in B$ such that for all $O_1 \subset O_x$ with $O_1 \in B$, O_1 is homeomorphic to O_x. We call the homeomorphism type of O_x the *local homeomorphism type of* x.

Thus, now letting $\xi(x) = O_x$, it encodes the local homeomorphism type around the point. In graphs this will be either a straight line or a node of some degree $d \in \mathbb{N}$. Metric realizations of graphs can have edges of varying length which should be ignored. One option is to use the topological history information equivalence (Example 5.9). The other option is to equip the complex first with a metric d in which all the nodes of degree d have isometric neighbourhoods and so that the distance between nodes equals one, and then redefine $d'(x, y) = \min\{d(x, y), \frac{1}{2}\}$ to lose all non-local information.

By Theorem 6.9, if the local homeomorphism type is all the robot can ever see, it cannot distinguish between 1-complexes which have the same universal covering.

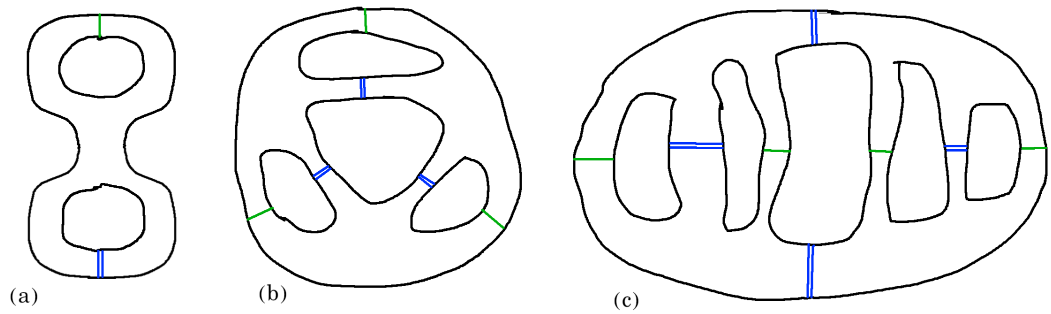

Figure 8: Indistinguishable environments for the beam-detecting robot. In each there are two types of beams, a single (green) and a double (blue) beam. All three environments are ≡-equivalent, and (b) and (c) are covering spaces of (a). Therefore, any arrangement of beams on (a) can be "lifted" to (b) and (c), making them indistinguishable.

However, it *can* distinguish between those that do not have. Most algorithms in this area will exploit other tools such as edge-labeling or pebble placing, for the robot to recognize which nodes have been visited already. ⊣

Open Problem 8.4. *Develop algorithms for a robot to distinguish between non-≡-equivalent 1-complexes without edge labeling or pebble placing.*

Open Problem 8.5. *Can our present theory elegantly accommodate the "pebble placing"? It seems that even a single pebble can significantly narrow down the space of possible worlds in which the robot could find itself.*

Example 8.6. Consider the example of beam sensing [28] of Figure 6. A robot moves in a multiply connected environment and whenever it crosses a beam, it senses the label of this beam. In [28] the authors show that under certain assumptions this helps the robot to determine homotopy invariants of its own trajectory. One of the assumptions is that each beam has a unique label. By dropping this assumption we can, using Corollary 6.7, design various environments which will be indistinguishable from the perspective of such a robot. We show some examples in Figure 8. To see that the spaces (b) and (c) are covering spaces of (a), we refer to [16, p. 58]. Note that by Corollary 6.7, *any* arrangement of beams in (a) can be lifted to an arrangement of beams in (b) and (c) so that the environments become indistinguishable. Whereas (b) and (c) are also ≡-equivalent (Theorem 6.9(B)), neither one is a covering of the other one; thus, not all arrangements of beams on (b) can be lifted to (c), or vice versa. Environments (b) and (c) are examples of equivalent ones between which there is no map witnessing the equivalence. Only a many-to-many bisimulation witnesses the equivalence (Theorem 7.4). ⊣

Motivated by the idea of local uniformity expressed in Examples 8.2 and 8.3, we can formulate a general theorem. We will formulate it for the metric uniformity, leaving topological uniformity for future work. Let P be the space of all Polish metric spaces (for example viewed as the Effros space of all closed subsets of the Urysohn space [11, 19]). For any Polish metric space $X \in P$ which is locally metrically uniform, let $\xi^X : X \to P$ be a function such that $\xi^X(x) = \bar{B}_X(x, \varepsilon_x)$ where ε_x is a witness for local metric uniformity at x. Since the set of such ε_x is a connected subset of $\mathbb{R}_+$ (since it is downward closed), it is K_σ and so ξ^X can always be chosen to be Borel by the Arsenin-Kunugui uniformization theorem [19, 18.18]. Note that since $\bar{B}_X(x, \varepsilon_x)$ is a closed subset of X, it is also a member of P. By a metric covering map f we mean a covering map which is a local isometry. Fix $g : P \to Y$ to be any isometry-invariant function which means that $g(M) \neq g(M')$ implies that M and M' are not isometric. Recall that $\mathcal{U}_M$ is the space of all measurable control signals (Definition 3.1). Below all path actions are assumed to have the domain $\mathcal{U}_M$.

Theorem 8.7. *Suppose X and X' are locally metrically uniform Polish spaces, $h = g \circ \xi^X$ and $h' = g \circ \xi^{X'}$, and that p and p' are initialized path actions on X and X' with $p(\varnothing) = x_0$ and $p'(\varnothing) = x'_0$. Suppose there is a metric covering map $f : (X', x'_0) \to (X, x_0)$ such that $f \circ p' = p$. Then, the environments (X, x_0, h, p) and (X', x'_0, h', p') are $\equiv$-equivalent.*

Proof. We need to show that $h' = h \circ f$. To see this, let $x' \in X'$. Let ε be small enough such that $f \restriction B(x', \varepsilon)$ is an isometry, and for all $\delta < \varepsilon$, $B_X(x', \delta)$ is isometric to $\xi^{X'}(x')$ and $B_{X'}(f(x), \delta)$ is isometric to $\xi^X(f(x'))$. However, an isometry takes balls to balls of the same radius; thus, $\xi^X(f(x'))$ is isometric to $\xi^{X'}(x')$ which implies $(g \circ \xi^{X'})(x') = (g \circ \xi^X)(f(x))$ by the property that g is invariant with respect to isometry. By the definition of h and h' this means $h'(x') = h(f(x))$. □

Corollary 8.8. *Suppose X and X' are locally metrically uniform Polish spaces, $h = g \circ \xi^X$ and $h' = g \circ \xi^{X'}$, and that p is an initialized path action on X with $p(\varnothing) = x_0$. Suppose there is a metric covering map $f : (X', x'_0) \to (X, x_0)$ for some $x'_0 \in X'$. Then there is a path action p' on X' such that the environments (X, x_0, h, p) and (X', x'_0, h', p') are $\equiv$-equivalent.*

Proof. Using Corollary 6.7, let p' be a lifting of p and apply Theorem 8.7. □

Open Problem 8.9. *Prove results for topological uniformity that are analogous to Theorem 8.7 and Corollary 8.8.*

9 Conclusion

This paper has formalized and unified previous notions pertaining to robots that explore unknown environments using limited sensors. We started by motivating a mathematical analysis of the robotics loop closure problem. Perhaps our approach best describes the situation of false positives in loop closure. Indeed, if $p(\bar{u}) \neq p'(\bar{u})$, but $h(p(\bar{u})) = h'(p'(\bar{u}))$, then we might be tempted to infer that loop closure was detected, but it is a false alarm. This false positive at the extreme is tantamount to the inability to distinguish between the environments completely, that is when $h \circ p = h' \circ p'$. Our intuition was that this is closely related to the idea of covering spaces because covering maps literally *close loops* by mapping, at best, contractible spaces onto spaces with non-trivial fundamental groups. With this motivation, we developed a general topological theory that relates control signals, trajectories, and path actions in Section 3. Building on the framework in [36] and the general theory of dynamical systems, we defined a continuous-time version of history information spaces. Then, we applied the resulting tools to define various equivalence relations on environments, which are of independent topological and set theoretic interest (recall Open Problem 5.14). We then moved on to the main motivation, covering spaces, and proved that if X' is a covering space of X, then indeed any sensorimotor structure can be lifted from X to X', making it look exactly the same from the point of view of a robot (Theorem 6.5). For this theorem we assumed that $\mathcal{U}$ is path-connected and locally path-connected. We proved this for the space of measurable controls $\mathcal{U}_M$ (Proposition 6.6), but we left open whether these conditions can be weakened, and by how much (Open Problem 6.8). Covering spaces and covering maps may be attractive from the point of view of applications as they preserve local structure and are convenient to work with, but they do not give a complete mathematical characterization of the indistinguishability relation. This is why in Section 7 we used the notion of bisimulation to obtain a necessary and sufficient condition for the equivalence to take place. The final result was Theorem 8.7 along with Corollary 8.8, which unifies and explains several robot navigation settings, by characterizing environment ambiguities in terms of covering spaces.

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