Pointers to Quasi-Monte Carlo Literature
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Many people have asked me about references to sampling literature that are relevant to motion planning, especially since the appearance of our WAFR and ICRA papers (with M. Branicky, K. Olson, L. Yang).

The first couple of chapters of the following book provide very clear coverage of low-discrepancy sequences:

```
@book{Mat99,
        author={J. Matousek},
        title={Geometric Discrepancy},
        publisher={Springer-Verlag},
        address={Berlin},
        year={1999}}
```

There are many nice illustrations.

The most complete reference on the subject, especially for topics relevant to robotics, is:

```
@book{Nie92,
        author = {H. Niederreiter},
        title = {Random Number Generation and Quasi-{M}onte-{C}arlo Methods},
        publisher = {Society for Industrial and Applied Mathematics},
        address = {Philadelphia, USA},
        year = 1992
        }
```

Chapter 6 is most relevant to motion planning because it deals with dispersion, whereas most QMC literature focuses on discrepancy.

There have been many interesting developments since Niedereiter's book. A recent snapshot of the field appears in Monte Carlo and Quasi-Monte Carlo Methods 2000, K.-T. Fang, F.J. Hickernell, H. Niederreiter, Eds., Springer-Verlag, 2002. In this book, there are some excellent surveys, including:

Quasi-Monte Carlo: The Discrepancy Between Theory and Practice, Shu Tezuka, pp. 124-140.

An Historial Overview of Lattice Point Sets, Y. Wang, F.J. Hickernell, pp. 158-167.

There are three general classes of low-discrepancy sequences/point sets:

1. Halton/Hammersley

2. Lattices (including grids)
3. (t,m,s)-nets and (t,s)-sequences

The first class represents a single sequence and point set. The second class usually represents point sets that have a regular neighborhood structure, and there is recent interest in infinite sequences, called extensible lattices. Most modern low-discrepancy sequences fall into the third class, which does not include the first two. The most famous in the third class are Sobol' and Faure sequences. The best known family in this class (and in general) are the Niederreiter-Xing sequences. Here is some software that generates them:

http://www.dismat.oeaw.ac.at/pirs/niedxing.html

Note that all classes above are concerned with low-discrepancy, and not necessarily low-dispersion, which we believe is more relevant for motion planning. In this case, low-dispersion grids (such as Sukharev) and extensible-grid sequences, have much better dispersion than any of the classes above. They can achieve this because they are not concerned with alignments, as required by minimizing discrepancy.

All three classes above are covered in Niederreiter's and Matousek's books.

Randomized versions of low-discrepancy sequences have been considered in recent years. Some randomized lattices and (t,m,s)-nets are covered in Strategies for Quasi-Monte Carlo, Bennet L. Fox, Kluwer, 1999.

This recent paper covers an elegant constructio for obtaining randomized Halton sequences:

X. Wang and F. J. Hickernell, Randomized Halton Sequences, Math. Comp. Modelling 32 (2000), pp. 887-899.

In general, the goal of randomized QMC is to make versions such that 1) each element of the sequence is uniformly, randomly distributed, and 2) all of the low-discrepancy/low-dispersion properties are preserved under the randomization.